Completeness of Bethe's states for generalized XXZ model, II.

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Abstract

For any rational number $p_0 \geq 2$ we prove an identity of Rogers-Ramanujan's type. Bijection between the space of states for XXZ model and that of XXX model is constructed.

The main goal of our paper is to study a combinatorial relationship between the space of states for generalized XXZ model and that for XXX one. In our previous paper [KL] we gave a combinatorial description of states for generalized XXZ model in terms of the so-called rigged $sl(2)$–XXZ configurations. On the other hand it is well-known that when the anisotropy parameter p_0 of XXZ model goes to infinity then the XXZ model under consideration transforms to the XXX one. We are going to describe this transformation from combinatorial point of view in the case when p_0 is an integer.

A combinatorial completeness of Bethe's states for generalized XXX-model was proven in [K1] and appears to be a starting point for numerous applications to combinatorics of Young tableaux and representation theory of symmetric and general linear groups, see e.g. [K2]. Here we mention only a "fermionic" formula for the Kostka-Foulkes polynomials, see e.g. [K2], and the relationship of the last with $\hat{sl}(2)$- branching functions $b_{\lambda}^{k\Lambda_0}(q)$, see

e.g. [K3]. We will show in §1, Theorem 2, that q-counting of the number of XXZ states using Bethe's ansatz approach [TS], [KR], gives rise to the Rogers-Ramanujan type formula for any rational number $p_0 > 2$.

It seems an interesting problem to find a polynomial version of the Rogers-Ramanujan type identity from our Theorem 2.

Another question which we are interested in is to understand a combinatorial nature of the limit

$$XXZ \underset{p_0 \to +\infty}{\longrightarrow} XXX.$$

In §2 we will describe a combinatorial rule which shows how the XXZ-configurations fall to the XXX pieces. For simplicity we consider in our paper only the case $p_0 > \sum_m s_m$. General case will be considered elsewhere.

§1. Rogers-Ramanujan's type identity.

This paper is a continuation of our previous work [KL]. Let us remind the main definitions, notations and results from [KL].

For fixed $p_0 \in \mathbf{R}$, $p_0 \geq 2$ let us define (cf. [TS]) a sequence of real numbers p_i and sequences of integer numbers ν_i, m_i, y_i:

$$p_0 := p_0,\ p_1 = 1,\ \nu_i = \left[\frac{p_i}{p_{i+1}}\right],\ p_{i+1} = p_{i-1} - \nu_{i-1} p_i,\ i = 1, 2, \dots \quad (1)$$

$$y_{-1} = 0,\ y_0 = 1,\ \ y_1 = \nu_0,\ y_{i+1} = y_{i-1} + \nu_i y_i,\ i = 0, 1, 2, \dots \quad (2)$$

$$m_0 = 0,\ m_1 = \nu_0,\ m_{i+1} = m_i + \nu_i,\ i = 0, 1, 2, \dots \quad (3)$$

$$r(j) = i,\ \text{if } m_i \leq j < m_{i+1},\ j = 0, 1, 2, \dots \quad (4)$$

It is clear that integer numbers ν_i define the decomposition of p_0 into a continuous fraction

$$p_0 = [\nu_0, \nu_1, \nu_2, \dots] = \nu_0 + \cfrac{1}{\nu_1 + \cfrac{1}{\nu_2 + \dots}}.$$

Let us define (see Fig. 1) a piecewise linear function n_j, $j \geq 0$,

$$n_j := y_{i-1} + (j - m_i) y_i, \quad \text{if} \ \ m_i \leq j < m_{i+1}. \quad (5)$$

It is clear that for any integer $n > 1$ there exists the unique rational number t such that $n = n_t$.

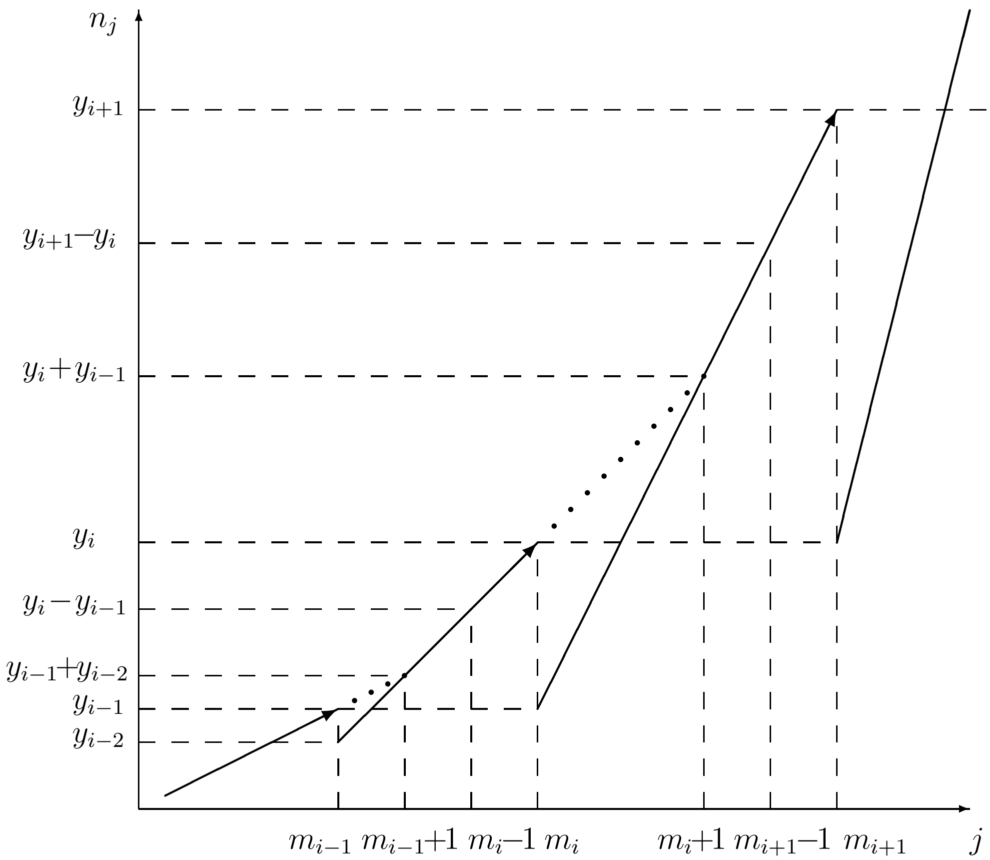


Fig.1. Image of piecewise linear function n_j in the interval $[m_{i-1}, m_{i+1}]$

Let us introduce additionally the following functions (see [KL])

$$q_j = (-1)^i (p_i - (j - m_i)p_{i+1}), \quad \text{if} \quad m_i \le j < m_{i+1}, \tag{6}$$

$$\Phi_{k,2s} = \begin{cases} \dfrac{1}{2p_0}(q_k - q_k n_\chi), & \text{if } n_k > 2s, \\ \dfrac{1}{2p_0}(q_k - q_\chi n_k) + \dfrac{(-1)^{r(k)-1}}{2}, & \text{if } n_k \le 2s, \end{cases}$$

where $2s = n_\chi - 1$.

In order to formulate our main result of part I about the number of Bethe's states for generalized XXZ model, let us consider the following symmetric matrix $\Theta^{-1} = (c_{ij})_{1\leq i,j\leq m_{\alpha+1}}$:

$$i) \qquad c_{ij} = c_{ji} \text{ and } c_{ij} = 0, \text{ if } |i-j| \geq 2.$$

$$ii) \qquad c_{j-1,j} = (-1)^{i-1}, \text{ if } m_i \leq j < m_{i+1}.$$

$$iii) \qquad c_{ij} = \begin{cases} 2(-1)^i, & \text{if } m_i \leq j < m_{i+1}-1, \quad i \leq \alpha, \\ (-1)^i, & \text{if } j = m_{\alpha+1}. \end{cases}$$

Example 1 For $p_0 = 4 + \dfrac{1}{5}$ one can find

$$\Theta^{-1} = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 1 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -1 & -1 \\ & & & & & & & -1 & 1 \end{pmatrix}$$

Further, let us consider a matrix $E = (e_{jk})_{1\leq j,k\leq m_{\alpha+1}}$, where

$$e_{jk} = (-1)^{r(k)}(\delta_{j,k} - \delta_{j,m_{\alpha+1}-1} \cdot \delta_{k,m_{\alpha+1}} + \delta_{j,m_{\alpha+1}} \cdot \delta_{k,m_{\alpha+1}-1}).$$

Then one can check that (cf. [KL], (3.9))

$$P_j(\lambda) + \lambda_j = ((E - 2\Theta)\widetilde{\lambda}^t + b^t)_j,$$

where $b = (b_1, \ldots, b_{m_{\alpha+1}})$ and

$$b_j = (-1)^{r(j)} \left(n_j \left\{ \frac{\sum 2s_m N_m - 2l}{p_0} \right\} - \sum_m 2\Phi_{j,2s_m} \cdot N_m \right).$$

Theorem 1 *(*[KL]*). The number of Bethe's states of generalized XXZ model, $Z^{XXZ}(N,s \mid l)$, is equal to*

$$\sum_{\lambda}\prod_{j}\left(\begin{array}{c} ((E-B)\widetilde{\lambda}^{t}+b^{t})_{j} \\ \lambda_{j} \end{array}\right), \tag{7}$$

where summation is taken over all configurations $\lambda=\{\lambda_k\}$ such that

$$\sum_{k=1}^{m_{\alpha+1}} n_k\lambda_k = l, \quad \lambda_k \geq 0;$$

$$\widetilde{\lambda}=(\widetilde{\lambda}_1,\ldots .\widetilde{\lambda}_{m_{\alpha+1}}), \quad \widetilde{\lambda}_j=(-1)^{r(j)}\lambda_j, \quad B=2\Theta.$$

One of the main goal of the present paper is to consider a natural q–analog for (7). Namely, let us define the following q–analog of the sum (7)

$$\sum_{\lambda} q^{\frac{1}{2}\widetilde{\lambda}B\widetilde{\lambda}^{t}}\prod_{j}\left[\begin{array}{c} ((E-B)\widetilde{\lambda}^{t}+b^{t})_{j} \\ \lambda_{j} \end{array}\right]_{q^{\epsilon_j}}, \tag{8}$$

where $\quad \epsilon_j=(-1)^{r(j)}$.

Let us remind that $\left[\begin{array}{c} M \\ N \end{array}\right]_q$ is the Gaussian q–binomial coefficient:

$$\left[\begin{array}{c} M \\ N \end{array}\right] = \begin{cases} \dfrac{(M)_q}{(N)_q(M-N)_q}, & \text{if } 0 \leq N \leq M, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Remark. In our previous paper [KL], see (5.1) and (5.2), we had considered another q-analog of (7). It turned out however that the q-analog (5.1) from [KL], probably, does not possess the good combinatorial properties.

One of the main results of Part II is the following

Theorem 2 *(*Rogers-Ramanujan's type identity*). Assume that p_0 be a rational number, $p_0 \geq 2$,and*

$$V_l(q)=\sum_{\lambda}\frac{q^{\widetilde{\lambda}B\widetilde{\lambda}^{t}}}{\prod_{j}(q;q^{\epsilon(j)})_{\lambda_j}}, \tag{9}$$

where summation in (9) is taken over all configurations $\lambda=\{\lambda_k\}$ *such that*

$$l=\sum_{k\geq 1} n_k\lambda_k,\quad \lambda_k\geq 0.$$

Then we have

$$\sum_{k\geq 0}(-1)^k q^{p_0k^2+\frac{k(k-1)}{2}}(1+q^k)=\sum_{l\geq 0} q^{\frac{l^2}{p_0}} V_l(q).$$

A proof is a "q-version" of that given in [KL], Theorem 4.1.

§2. $XXZ\to XXX$ bijection.

In this section we are going to describe a bijection between the space of states for XXZ-model and that of XXX-model. Let us formulate the corresponding combinatorial problem more exactly. First of all as it follows from the results of our previous paper, the combinatorial completeness of Bethe's states for XXZ model is equivalent to the following identity

$$\prod_m (2s_m+1)^{N_m}=\sum_{l=0}^{N} Z^{XXZ}(N,s\mid l), \tag{10}$$

where $N=\sum\limits_m 2s_mN_m$ and $Z^{XXZ}(N,s\mid l)$ is given by (7). On the other hand it follows from the combinatorial completeness of Bethe's states for XXX model (see [K1]) that

$$\prod_m (2s_m+1)^{N_m}=\sum_{l=0}^{\frac{1}{2}N}(N-2l+1)Z^{XXX}(N,s\mid l), \tag{11}$$

where $Z^{XXX}(N,s\mid l)$ is the multiplicity of $\left(\frac{N}{2}-l\right)$-spin irreducible representation of $sl(2)$ in the tensor product

$$V_{s_1}^{\otimes N_1}\otimes\cdots\otimes V_{s_m}^{\otimes N_m}.$$

Let us remark that both $Z^{XXZ}(N,s\mid l)$ and $Z^{XXX}(N,s\mid l)$ admits a combinatorial interpretation in terms of rigged configurations. The difference between the space of states of XXX model and that of XXZ model is the

availability of the so-called 1^--configurations (or 1^- string) in the space of states for the last model. The presence of 1^--strings in the space of states for XXZ-model is a consequence of broken $sl(2)$-symmetry of the XXZ-model. Our goal in this section is to understand from a combinatorial point of view how the anisotropy of XXZ model breaks the symmetry of the XXX chain. More exactly, we suppose to describe a bijection between XXZ-rigged configurations and XXX-rigged configurations. Let us start with reminding a definition of rigged configurations.

We consider at first the case of $sl(2)$ XXX-magnet. Given a composition $\mu = (\mu_1, \mu_2, \ldots)$ and a natural integer l by definition a $sl(2)$-configuration of type (l, μ) is a partition $\nu \vdash l$ such that all vacancy numbers

$$P_n(\nu; \mu) := \sum_k \min(n, \mu_k) - 2\sum_{k \le n} \nu'_k \tag{12}$$

are nonnegative. Here ν' is the conjugate partition. A rigged configuration of type (l, μ) is a configuration ν of type (l, μ) together with a collection of integer numbers $\{J_\alpha\}_{\alpha=1}^{m_n(\nu)}$ such that

$$0 \le J_1 \le J_2 \le \cdots \le J_{m_n(\nu)} \le P_n(\nu; \mu).$$

Here $m_n(\nu)$ is equal to the number of parts equal to n of the partition ν. It is clear that total number of rigged configurations of type (l, μ) is equal to

$$Z(l \mid \mu) := \sum_{\nu \vdash l} \prod_{n \ge 1} \left(\begin{array}{c} P_n(\nu; \mu) + m_n(\nu) \\ m_n(\nu) \end{array} \right).$$

The following result was proven in [K1].

Theorem 3 *Multiplicity of $(N - 2l + 1)$-dimensional irreducible representation of $sl(2)$ in the tensor product*

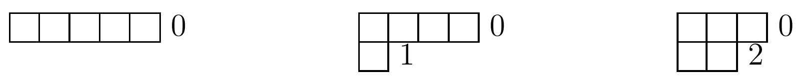
$$V_{s_1}^{\otimes N_1} \otimes \cdots \otimes V_{s_m}^{\otimes N_m}$$

is equal to $Z\left(l \mid \underbrace{2s_1, \ldots, 2s_1}_{N_1}, \ldots, \underbrace{2s_m, \ldots, 2s_m}_{N_m}\right)$.

Example 2 One can check that

$$V_1^{\otimes 5} = 6V_0 + 15V_1 + 15V_2 + 10V_3 + 4V_4 + V_5.$$

In our case we have $\mu = (2^5)$. Let us consider $l = 5$. It turns out that there exists three configurations of type $(3, (2^5))$, namely



Hence $Z(3 \mid (2^5)) = 1 + 2 + 3 = 6 = \mathrm{Mult}_{V_0}\left(V_1^{\otimes 5}\right)$.

Now let us give a definition of $sl(2)$-XXZ configuration. We consider in our paper only the case when the anisotropy parameter p_0 is an integer, $p_0 \in \mathbf{Z}_{\geq 2}$. Under this assumption the formulae (5) and (6) take the following form:

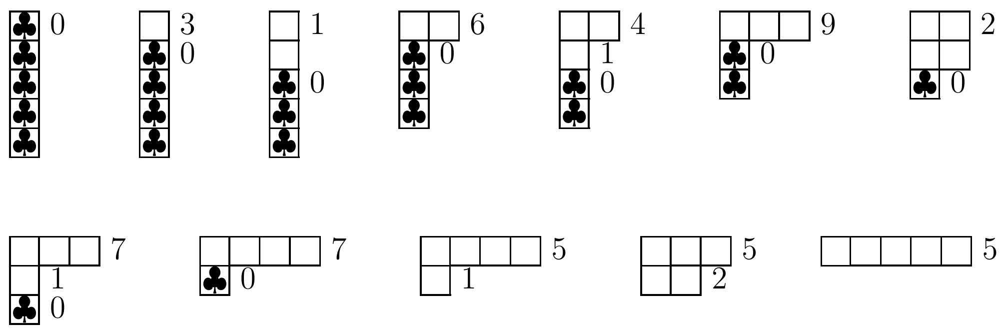
$$\begin{aligned}
&n_j = j, \text{ if } 1 \leq j < p_0, \ v_j = +1;\\
&n_{p_0} = 1, \ v_{p_0} = -1;\\
&2\Phi_{k,2s} = \frac{2sk}{p_0} - \min(k, 2s), \text{ if } 1 \leq k < p_0, \ 2s + 1 < p_0;\\
&2\Phi_{p_0,2s} = \frac{2s}{p_0}, \text{ if } 2s + 1 < p_0;\\
&b_{kj} = k - j, \text{ if } 1 \leq j \leq k < p_0;\\
&b_{kp_0} = 1, \text{ if } 1 \leq k < p_0;\\
&a_j := a_j(l \mid \mu) = \sum_m \min(j, \mu_m) - 2l - j\left[\frac{\sum_m \mu_m - 2l}{p_0}\right], \text{ if } 1 \leq j < p_0;\\
&a_{p_0}(l \mid \mu) = \left[\frac{\sum_m \mu_m - 2l}{p_0}\right].
\end{aligned}$$

Definition 1 *A $sl(2)$-XXZ-configuration of type (l, μ) is a pair (λ, λ_{p_0}), where λ is a composition with all parts strictly less than p_0, $\sum_{j<p_0} j\lambda_j + \lambda_{p_0} = l$, and such that all vacancy numbers $P_j(\lambda \mid \mu)$ are nonnegative.*

Let us remind ([KL]) that

$$P_j(\lambda|\mu) := a_j(l \mid \mu) + 2\sum_{j<k<p_0}(k-j)\lambda_k + \lambda_{p_0}, \text{ if } j < p_0 - 1; \quad (13)$$
$$P_{p_0-1}(\lambda \mid \mu) := a_{p_0-1}(l \mid \mu) + \lambda_{p_0};$$
$$P_{p_0}(\lambda \mid \mu) := a_{p_0}(l \mid \mu) + \lambda_{p_0-1}.$$

Example 3 Let us consider $p_0 = 6$, $s = \frac{3}{2}$, $N = 5$, $l = 5$. The total number of type $(5,(3^5))$ $sl(2)$–XXZ configurations is equal to 12.

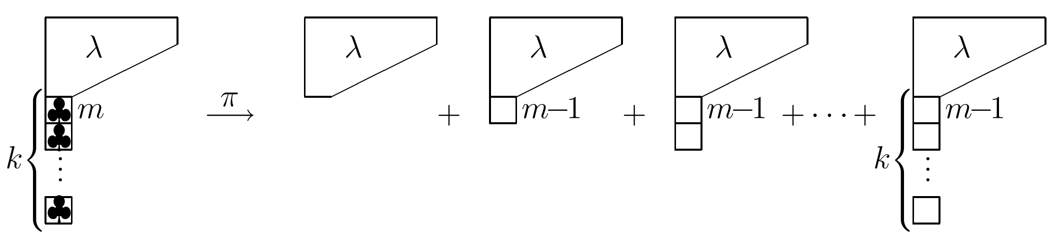


The total number of type $(5,(3^5))$ rigged configurations is equal to

$$Z^{XXZ}(5 \mid (3^5)) = 101 = 1+4+3+7+10+10+6+16+8+12+18+6.$$

Here we used a symbol $\clubsuit$ to mark a 1^-–strings.

Now we are ready to describe a map from the space of states for XXZ model to that of XXX one. More exactly we are going to describe a rule how a XXZ-configuration fall to the XXX-pieces. At first we describe this rule schematically:



This decomposition corresponds to the well-known identity

$$\left[\begin{array}{c} m+k \\ k \end{array}\right]_q = \sum_{j=0}^{k} q^j \left[\begin{array}{c} m+j-1 \\ j \end{array}\right]_q .$$

In what follows we will assume that $p_0 > \sum_m s_m$.

Theorem 4 *The map π is well-defined and gives rise to a bijection between the space of states of XXZ-model and that of XXX one.*

Proof. Let us start with rewriting the formulae (13) for the XXZ-vacancy numbers in more convenient form, namely,

$$P_j^{XXZ}(\widetilde{\nu} \mid \mu) = \sum_m \min(j, 2s_m) - 2\sum_{k\leq j} \nu'_k - j\left[\frac{\sum_m 2s_m - 2l}{p_0}\right],$$
if $1 \leq j < p_0 - 1$;

$$P_{p_0-1}^{XXZ}(\widetilde{\nu} \mid \mu) = p_0\left\{\frac{\sum_m 2s_m - 2l}{p_0}\right\} + \left[\frac{\sum_m 2s_m - 2l}{p_0}\right] + \lambda_{p_0}; \qquad (14)$$

$$P_{p_0}^{XXZ}(\widetilde{\nu} \mid \mu) = \left[\frac{\sum_m 2s_m - 2l}{p_0}\right] + m_{p_0-1}(\nu).$$

Here $\mu = (2s_1, \ldots, 2s_m)$ and $\widetilde{\nu}$ is a pair $\widetilde{\nu} = (\nu, \lambda_{p_0})$, where ν is a partition such that $l(\nu) \leq p_0 - 1$, $|\nu| + \lambda_{p_0} = l$. Relationship between λ from Definition 1 and ν is the following

$$m_j(\nu) = \lambda_j, \quad \text{i.e.} \quad \nu = (1^{\lambda_1} 2^{\lambda_2} \ldots (p_0-1)^{\lambda_{p_0-1}}).$$

Now let us consider an integer $l \leq \sum_m s_m$ and let $\nu \vdash l$ be a XXX-configuration. Let λ_{p_0} be integer such that $2\sum s_m - 2l - p_0 < \lambda_{p_0} \leq \sum_m s_m - l$ and consider the pair $\widetilde{\nu} = (\nu, \lambda_{p_0})$. It is easy to check that

$$P_j^{XXZ}(\widetilde{\nu} \mid \mu) = \sum_m \min(j, 2s_m) - 2\sum_{k\leq j} \nu'_k = P_j^{XXX}(\nu \mid \mu) \geq 0,$$
if $1 \leq j < p_0 - 1$;

$$P_{p_0-1}^{XXZ}(\widetilde{\nu} \mid \mu) = \sum_m 2s_m - 2l + \lambda_{p_0} \geq 0;$$

$$P_{p_0}^{XXZ}(\widetilde{\nu} \mid \mu) = \lambda_{p_0-1} \geq 0.$$

Thus the pair $\widetilde{\nu} = (\nu, \lambda_{p_0})$ is a XXZ-configuration.

Furthermore it follows from our assumptions (namely, $\sum_m s_m < p_0$, $\lambda_{p_0} > 0$) that $\lambda_{p_0-1} = 0$ and both 1^--strings and $(p_0 - 1)$-strings do not give a contribution to the space of XXZ-states. Thus we see that both XXX-configuration ν and XXZ-configuration $\widetilde{\nu} = (\nu, \lambda_{p_0})$ defines the same number of states. Now, if $\widetilde{\nu} = (\nu, \mu)$ is a XXZ-configuration then ν is a XXX configuration as well. This is clear because (see (14))

$$P_j^{XXX}(\nu \mid \mu) \geq P_j^{XXZ}(\widetilde{\nu} \mid \mu), \quad 1 \leq j \leq p_0 - 1.$$

By the similar reasons if $(\widetilde{\nu}, \lambda_{p_0})$ is a XXZ-configuration then for any $0 \leq k \leq \lambda_{p_0}$ the pair $(\widetilde{\nu}, \lambda_{p_0} - k)$ is also XXZ-configuration. It follows from what we say above that π is the well-defined map. Furthermore there exists one to one correspondence between the space of XXX-configurations and that of XXZ-configurations, namely,

$$\nu \leftrightarrow \widetilde{\nu} = (\nu, \lambda_{p_0}),$$

where $\lambda_{p_0} = [\sum_m s_m - |\nu|]$.

All others XXZ-configurations (ν, k) with $0 \leq k < \sum_m s_m - |\nu| - 1$ give a contribution to the space of descendants for $\nu \leftrightarrow \widetilde{\nu}$.

$\blacksquare$

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